Tan Relations in Dilute Bose Gasses

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The exact relations for strongly interacting Fermi gasses, recently derived by Tan, are shown to first order in the loop expansion to also apply to dilute Bose gasses. A simple thermodynamic argument is put forward to support their validity. As an application, the second-order correction to the depletion of the condensate is determined to logarithmic accuracy.

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In a sequence of profound original papers [1–3], Tan recently derived various exact relations for strongly correlated Fermi gasses. Generalizing the standard formula for a free Fermi gas, he expressed the ground-state energy of a (stongly) interacting Fermi gas with two spin states as a simple linear functional of the particle momentum distribution $n(k)$, which he argued to fall off like C/k^4 at large wave number k. Finiteness of the energy in the ultraviolet requires that this tail be subtracted. Tan demonstrated that the ground-state energy explicitly depends on the coefficient C of this tail. He, moreover, showed that this coefficient, the "contact", determines the change in ground-state energy due to a small variation in the s-wave scattering length, which he dubbed "the adiabatic sweep theorem", and that it features in a relation connecting the pressure to the ground-state energy as well as in a generalized virial theorem. In essence, these results imply that the sole parameter C captures the correlations of a Fermi gas (with short-range interactions). It is remarkable that this parameter, characterizing the behavior of interacting fermions at *short* distances ($k \to \infty$), features in thermodynamic quantities that characterize the system at *long* distances. Two of Tan's predictions, viz. the adiabatic sweep and the generalized virial theorems, were very recently verified in experiments at JILA on ultra cold gases of fermionic ^{40}K atoms confined in a harmonic trapping potential [4].

It was pointed out by Combescot *et al.* [5] that the statistics of the particles plays no role in the derivation of the Tan relations, so that they should also apply to Bose gasses. In this note, we explicitly show this to be the case to first order in the loop expansion. The exactness of the relations is supported by a simple thermodynamic argument we put forward. Using these relations, we determine, for the first time, the second-order correction to the depletion of the condensate to logarithmic accuracy.

Adapted for a dilute Bose gas of one species of particles with number density n, the Tan energy relation [1] becomes [5]

$$\mathcal{E}_{\rm T}(n) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \epsilon(k) \left[n(k) - \frac{C}{k^4}\right] + \frac{\hbar^2 C}{8\pi m a}. \tag{1}$$

Here, $\epsilon(k) = \hbar^2 k^2/2m$ is the spectrum of *noninteracting* bosons and a is the s-wave scattering length characterizing the short-range, but not necessarily weak interactions. The additional factor of $\frac{1}{2}$ in the last term in comparison to the corresponding term for an interacting Fermi gas with two spin states arises because only one species of particles is present here.

Equation (1) is to be compared to the standard perturbative expression for the ground-state energy density at the absolute zero of temperature [6],

$$\mathcal{E}(n) = \frac{2\pi\hbar^2 a n^2}{m} + \frac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3} \left[E(k) - \epsilon(k) - g n_0 + \frac{g^2 n_0^2}{2\epsilon(k)}\right], \tag{2}$$

obtained to first order in the loop expansion. Instead of the free particle spectrum, this expression dominantly features the Bogoliubov spectrum [7] $E(k) = \sqrt{\epsilon^2(k) + 2 g n_0 \epsilon(k)}$, where g is the bare coupling constant of the local interaction term in the Hamiltonian density

$$\mathcal{H}_{\rm int} = \frac{1}{2} g \left(\psi^* \psi\right)^2, \tag{3}$$

with ψ the quantum field describing the atoms. Moreover, n_0 denotes the number density of particles in the condensate, which to lowest order is given by $n_0 = n$. Since the integral in Eq. (2) represents a one-loop contribution, n_0 in the integrand may to this order be replaced with n, so that the right side becomes a function of n alone. The first term in the integrand is the zero-point energy of the quantum field describing the Bogoliubov excitations. This contribution to the ground-state energy diverges in the ultraviolet and must be regularized. The natural way to do so in the context of Bose gasses, and in line with the modern appoach to renormalization group theory, is by introducing a large wave number cutoff Λ. This parameter physically denotes the scale beyond which the microscopic model (3) seizes to be valid. The effect of the unknown physics above the cutoff is incorporated by redefining, or renormalizing the parameters of the original theory so that physical observables become finite when expressed in terms of these renormalized parameters. The remaining three so-called counter terms in the integrand of Eq. (2), which render the integral finite in the limit $\Lambda \to \infty$, serve this purpose. The first counter term, which subtracts the zero-point energy of a free boson field, amounts to an irrelevant additive constant $\propto \Lambda^5$ independent of the parameters of the theory. (In nonrelativistic

theories, the mass parameter is just an atomic constant as far as renormalization is concerned.) The second and third counter terms renormalize respectively the chemical potential, which has been swapped for the particle number density in Eq. (2), and the coupling constant [8]. Specifically,

$$\mu_{\rm r} \equiv \mu - \frac{1}{12\pi^2} g\Lambda^3, \tag{4a}$$

$$\begin{aligned} g_{\rm r} &\equiv g - g^2 \int \frac{{\rm d}^3 k}{(2\pi)^3} \frac{1}{2\epsilon(k)} \\ &= g - \frac{1}{2\pi^2}\frac{m}{\hbar^2} g^2 \Lambda, \end{aligned} \tag{4b}$$

where renormalized parameters are given the subscript "r". The renormalized coupling constant is related to the s-wave scattering length a through $g_{\rm r} = 4\pi\hbar^2 a/m$. This has been used in the first term at the right side of Eq. (2), $\frac{1}{2} g_{\rm r} n^2$, which represents the contribution from the condensate. Evaluation of the wave vector integrals in Eq. (2) yields in the limit $\Lambda \to \infty$ the celebrated result due to Lee and Yang [9]

$$\mathcal{E}(n) = \frac{2\pi\hbar^2 a n^2}{m}\left[1 + \frac{128}{15}\left(\frac{a^3 n}{\pi}\right)^{1/2}\right], \tag{5}$$

where, consistent to one-loop oder, also g in the one-loop integral has been replaced with $g_{\rm r}$.

As an aside, we remark that in dimensional regularization no counter terms are needed at all in Eq. (2), for positive powers of the cutoff do not show up in this scheme [8].

The particle momentum distribution $n(k)$ featuring in Tan's energy relation (1) reads to this order [6]

$$n(k) = \frac{1}{2}\left(\frac{\epsilon(k) + g n_0}{E(k)} - 1\right). \tag{6}$$

It varies at large wave numbers in accordance with Tan's observation as $n(k) \sim C_0/k^4$, with $C_0 \equiv \left(m g n_0/\hbar^2\right)^2$. Upon substituting this for C in the integrand of Eq. (1), which is justified to one-loop order, and carrying out the wave vector integrals, we obtain in the limit $\Lambda \to \infty$

$$\mathcal{E}_{\rm T}(n) = -\frac{128\pi^{1/2}}{5}\frac{\hbar^2 (an)^{5/2}}{m} + \frac{\hbar^2 C}{8\pi m a}. \tag{7}$$

To demonstrate that this agrees with the Lee-Yang result (5), we compute the contact C to first order in the loop expansion.

To this end, we use the result derived by Braaten and Platter [10] through the operator product expansion that the contact is determined by the expectation value of the interaction term (3)

$$C = \left(\frac{mg}{\hbar^2}\right)^2 \left\langle (\psi^*\psi)^2 \right\rangle. \tag{8}$$

This result was originally derived for a Fermi gas, but since the derivation is independent of the statistics of the particles, it also applies here. To account for the condensate, the (complex) field ψ is shifted as follows

$$\psi = v + \frac{1}{\sqrt{2}}\left(\chi_1 + {\rm i}\chi_2\right), \tag{9}$$

with $\chi_1(t,\mathbf{x})$ and $\chi_2(t,\mathbf{x})$ two real fields. The expectation value v of the field ψ is related to the number density n_0 of condensed particles through $v^2 = n_0$. Without loss of generality, we assumed that v is real. The propagator of the multiplet $(\chi_1, \chi_2)^{\rm T}$ as a function of frequency ω and wave vector $\mathbf{k}$ is given at lowest order by

$$\Delta_{\rm F}(\omega, \mathbf{k}) = \frac{1}{\hbar^2\omega^2 - E^2(\mathbf{k}) + {\rm i}\eta}\begin{pmatrix} \epsilon(\mathbf{k}) & {\rm i}\hbar\omega \\ -{\rm i}\hbar\omega & \epsilon(\mathbf{k}) + 2 g n_0 \end{pmatrix}, \tag{10}$$

with $\eta > 0$ the usual infinitesimal parameter, introduced for causality, that is to be set to zero at the end of the calculation, and $E(\mathbf{k})$ the Bogoliubov spectrum. To the one-loop order, $\left\langle (\psi^*\psi)^2 \right\rangle = n_0^2 + 3 n_0 \left\langle \chi_1^2 \right\rangle + n_0 \left\langle \chi_2^2 \right\rangle$ so that by Eq. (10)

$$C = \left(\frac{mg}{\hbar^2}\right)^2 n_0 \left(n_0 + \frac{3}{2}\int \frac{{\rm d}^3 k}{(2\pi)^3}\frac{\epsilon(k)}{E(k)} + \frac{1}{2}\int \frac{{\rm d}^3 k}{(2\pi)^3}\frac{E(k)}{\epsilon(k)}\right) \tag{11}$$

after carrying out the frequency integrals. Note that the wave vector integrals appearing here diverge in the ultraviolet. Apart from an irrelevant diverging term $\propto \Lambda^3$ independent of the parameters of the theory, the divergences can be canceled as in the fermionic theory [10] by going over to the renormalized coupling constant, giving

$$C = \left(\frac{m g_{\rm r} n_0}{\hbar^2}\right)^2 \left[1 + \frac{80}{3}\left(\frac{a^3 n}{\pi}\right)^{1/2}\right]. \tag{12}$$

The density of particles n_0 residing in the condensate is given to this order by the Bogoliubov depletion formula [7]

$$\begin{aligned} n_0 &= n - \int \frac{\mathrm{d}^3 k}{(2\pi)^3} n(k) \\ &= n\left[1 - \frac{8}{3}\left(\frac{a^3 n}{\pi}\right)^{1/2}\right]. \end{aligned} \tag{13}$$

Use of this in Eq. (12) yields for the contact to first order in the loop expansion

$$C = (4\pi a n)^2 \left[1 + \frac{64}{3}\left(\frac{a^3 n}{\pi}\right)^{1/2}\right]. \tag{14}$$

Inserting this expression for C into Eq. (7), we recover the Lee-Yang result (5) and thereby showed that, to first order in the loop expansion, the Tan energy relation (1) agrees with the Bogoliubov theory.

Given the result (14) for the contact, it is readily verified that Tan's adiabatic and pressure relations [2],

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}a} = \frac{\hbar^2 C}{8\pi m a^2}, \qquad P = \frac{2}{3}\mathcal{E} + \frac{\hbar^2 C}{24\pi m a}, \tag{15}$$

are satisfied by the standard expressions obtained from Bogoliubov theory. The last terms in these expressions carry again an additional factor of $\frac{1}{2}$ in comparison to their fermionic counterparts because only one species of particles is present here. The two (nonperturbative) relations (15) can be combined to yield for the pressure

$$P = \frac{1}{3a}\frac{\mathrm{d}}{\mathrm{d}a}\left(a^2 \mathcal{E}\right). \tag{16}$$

This expression is closely related to the thermodynamic relation

$$P = n^2 \frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{\mathcal{E}}{n}\right) \tag{17}$$

and can in fact be derived from it by noting that the leading term in the ground-state energy is proportional to an^2, while the quantum corrections are, on dimensional grounds, a function of $a^3 n$ alone. This simple, yet nonperturbative argument gives strong support to the validity of the Tan relations (15).

We next extend the analysis to second order in the loop expansion. The second-order expression for the ground-state energy [11–13] yields, to logarithmic accuracy, the extra term $(32/3)\left(4\pi - 3\sqrt{3}\right) a^3 n \ln(a^3 n)$ within the square brackets in Eq. (14). The calculation of the expectation value in Eq. (8) involves, apart from minor changes in the vertices, the same diagrams that contribute to the thermodynamic potential, which have been treated in detail in Ref. [14]. To logarithmic accuracy, we find

$$\left\langle (\psi^* \psi)^2 \right\rangle = n_0^2 \left[1 + \frac{80}{3}\left(\frac{a^3 n}{\pi}\right)^{1/2} + \frac{16}{3}\left(4\pi - 3\sqrt{3}\right) a^3 n \ln(a^3 n)\right]. \tag{18}$$

Inserting these results into Eq. (8), we obtain as next-order correction to the depletion of the condensate

$$n_0 = n\left[1 - \frac{8}{3}\left(\frac{a^3 n}{\pi}\right)^{1/2} + \frac{8}{3}\left(4\pi - 3\sqrt{3}\right) a^3 n \ln(a^3 n)\right]. \tag{19}$$

A direct calculation of this correction term, without using Tan relations, appears laborious and has, to our knowledge, not been reported in the literature.

At the face of it, Tan's energy relation (1) is surprising from the theory side in that it makes no explicit reference to the condensate. This is in sharp contrast to the standard perturbative approach where the condensate is explicitly accounted for from the onset, see Eq. (2). Given the experimental success in verifying the Tan relations in strongly interacting Fermi gasses, a similar experimental undertaking in the context of dilute Bose gasses seems desirable to gain further insights into strongly interacting systems.

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